APPENDIX C

ANALYTICAL SOLUTION FOR THREE-DIMENSIONAL STEADY-STATE GROUNDWATER FLOW IN A CONSTANT THICKNESS AQUIFER

Notation

B Thickness of aquifer (held constant) [L].

F Net leachate rate under the patch, expressed as a Darcian velocity.

$\left[\frac{L^3W}{L^2T}\right]$ Note that the net infiltration rate at any point is the sum of F and I.

K_y Hydraulic head [L].

H_1, H_2 Hydraulic head specified at the upstream (x=0) and downstream (x=L) boundaries [L]. These are boundary conditions for the x domain.

I Net regional infiltration rate, expressed as a Darcian velocity $\left[\frac{L^3W}{L^2T}\right]$.

K_x, K_y, K_z Saturated hydraulic conductivity $\left[\frac{L^3W}{L^2T}\right]$.

L Distance between the upstream and downstream specified heads [L].

x, y, z Spatial coordinates, where z is the vertical dimension, with the aquifer surface specified at z=0 and aquifer base as z=B [L].

x_1, x_2, y_1, y_2 Spatial coordinates defining the areal patch over which flux F is applied [L]. Note that $L \geq x_2 > x_1 \geq 0$ and $y_2 > y_1$.

Steady-state, 3-D flow in an aquifer is defined by Laplace's Equation

$$K_x \frac{\partial^2 H}{\partial x^2} + K_y \frac{\partial^2 H}{\partial y^2} + K_z \frac{\partial^2 H}{\partial z^2} = 0 \left[\frac{L_W^3}{L^3T}\right] \qquad \textbf{(C.1)}$$

with boundary conditions

$$
\begin{aligned}
&H(0,y,z,\infty) = H_1 \\
&H(L,y,z,\infty) = H_2 \\
&\frac{\partial H}{\partial y}\ (x,\pm\infty,z,\infty) = 0 \\
&-K_z\ \frac{\partial H}{\partial z}(x,y,0,\infty) \ = \ F[U(x-x_1)-U(x-x_2)][U(y-y_1) \ - \ U(y-y_2)] \ + \ I \\
&\frac{\partial H}{\partial z}(x,\, y,\, B,\infty) \ = \ 0
\end{aligned}
\tag{C.2}
$$

where I is the net Darcian infiltration rate of rainfall (uniformly constant), F is the net Darcian infiltration rate of leachate (applied only over the surface patch defined by x_1, x_2, y_1, y_2) and U (•) is the Heaviside unit step function.

A fundamental assumption of the above is that the saturated thickness B remains constant, despite the fact that there is mounding.

Consider the following integral transform for a finite x domain which has two first type boundary conditions:

$$\bar{H}(\beta_m, y, z, \infty) \ = \ \sqrt{\frac{2}{L}} \int_{x'=0}^{L} \sin\ (\beta_m x')\ H(x', y, z, \infty) dx' \tag{C.3}$$

and its inversion transform

$$H(x,y,z,\infty) \ = \ \sum_{m=1}^{\infty} \sqrt{\frac{2}{L}}\ \sin(\beta_m x) \bar{H}(\beta_m, y,\, z, \infty) \tag{C.4}$$

where the eigenvalues β_m are defined by

$$\beta_m \ = \ \frac{m\pi}{L} \qquad\qquad m \ = \ 1,2,\cdots \tag{C.5}$$

The integral transform for an infinite y domain is given by:

$$\bar{\bar{H}}(\beta_m, v, z, \infty) = \int_{y'=-\infty}^{\infty} e^{-ivy'} \bar{H}(\beta_m, y', z, \infty)dy' \quad \textbf{(C.6)}$$

and the inversion formula

$$\bar{H}(\beta_m, y, z, \infty) = \frac{1}{2\pi}\int_{v=-\infty}^{\infty} e^{-ivy'} \bar{\bar{H}}(\beta_m, v, z, \infty)dv \quad \textbf{(C.7)}$$

The integral transform for the finite z domain which has two second type boundary conditions is:

$$\bar{\bar{\bar{H}}}(\beta_m, v, \psi_n, \infty) = \frac{A_n}{\sqrt{B}}\int_{z'=0}^{\infty} \cos(\psi_n z')\, \bar{\bar{H}}(\beta_m, v, z', \infty)dz' \quad \textbf{(C.8)}$$

and the inversion formula

$$\bar{\bar{H}}(\beta_m, v, z, \infty) = \sum_{n=0}^{\infty} \frac{A_n}{\sqrt{B}} \cos(\psi_n z)\, \bar{\bar{\bar{H}}}(\beta_m, v, \psi_n, \infty) \quad \textbf{(C.9)}$$

where the coefficient A_n equals

$$A_n = \begin{cases} 1 & n = 0 \\ \sqrt{2} & n = 1, 2, \cdots \end{cases} \quad \textbf{(C.10)}$$

and eigenvalues ψ_n

$$\psi_n = \frac{\pi n}{B} \qquad n = 0, 1, 2, \cdots \quad \textbf{(C.11)}$$

Remove the x variation in Equation (C.1) by multiplying this equation by

$\sqrt{\frac{2}{L}}\ \sin\ (\beta_m x')$ and then

integrating the resultant expression with respect to x' from 0 to L, using the transform given by

Equation (C.3) to get:

$$\left\{K_x \sqrt{\frac{2}{L}}\ \sin\ (\beta_m x')\ \frac{\partial H}{\partial x}\right\}\Big|_0^L - \left\{K_x \beta_m \sqrt{\frac{2}{L}}\ \cos\ (\beta_m x)\ H\right\}\Big|_0^L - K_x \beta_m^2 \bar{H} \qquad \textbf{(C.12)}$$

$$+\ K_y \frac{\partial^2 \bar{H}}{\partial y^2} + K_z \frac{\partial^2 \bar{H}}{\partial z^2} = 0$$

The first term $K_x \sqrt{\frac{2}{L}}\ \sin\ (\beta_m x')\ \frac{\partial H}{\partial x}$ is equates to 0 when integrated from 0 to L. Substitute the boundary conditions of Equation (C.2) into Equation (C.12) and rearrange to get

$$-K_x \beta_m \sqrt{\frac{2}{L}}\ [H_2\ (-1)^m - H_1] - K_x \beta_m^2\ \bar{H} + K_y \frac{\partial^2 \bar{H}}{\partial x^2} + K_z \frac{\partial^2 \bar{H}}{\partial x_2} = 0 \qquad \textbf{(C.13)}$$

with boundary conditions

$$\frac{\partial \bar{H}}{\partial y}\ (\beta_m, \pm\infty, z, \infty) = 0$$

$$\frac{\partial \bar{H}}{\partial z}\ (\beta_m, y, B, \infty) = 0 \qquad \textbf{(C.14)}$$

$$-K_z \frac{\partial \bar{H}}{\partial z}(\beta_m, y, 0, \infty) = F\sqrt{\frac{2}{L}}\ [U(y-y_1) - U(y-y_2)] \int_{x'=x_1}^{x^2} \sin\ (\beta_m x')\ dx'$$

$$+\ I\sqrt{\frac{2}{L}} \int_{x'=0}^{L} \sin\ (\beta_m x') dx'.$$

Remove the y variation in Equation (C.13) by multiplying this Equation by $e^{ivy'}$ and then integrating the resultant expression with respect to y' from ±∞ using the transform given Equation (C.6) to get:

$$-K_x\beta_m\sqrt{\frac{2}{L}}\,[H_2\,(-1)^m - H_1]\int_{-\infty}^{\infty} e^{ivy'}dy' - K_x\beta_m^2\bar{\bar{H}} - K_y v^2\bar{\bar{H}} + K_z\frac{\partial^2\bar{\bar{H}}}{\partial z^2} = 0 \qquad \textbf{(C.15)}$$

with boundary conditions

$$\frac{\partial\bar{\bar{H}}}{\partial z}\,(\beta_m, v, B, \infty) = 0$$

$$K_z\frac{\partial\bar{\bar{H}}}{\partial z}\,(\beta_m, v, 0, \infty) = F\sqrt{\frac{2}{L}}\int_{x'=x_1}^{x_2}\sin\,(\beta_m x')\int_{y'=y_1}^{y2} e^{ivy'}dy'dx' + I\sqrt{\frac{2}{L}}\,\Big|_{x'=0}^{L}\sin\,(\beta_m x \qquad \textbf{(C.16)}$$

$$\bullet\int_{y'=-\infty}^{\infty} e^{ivy'}dy'\;dx'.$$

Remove the z variation in Equation (C.15) by multiplying this Equation by

$\frac{A_n}{\sqrt{B}}\cos\,(v_n z')$ and then

integrating the resultant expression with respect to z' from 0 to B, using the transform given by Equation (C.8) to get:

$$\begin{aligned}
&-K_x\beta_m\frac{A_n}{\sqrt{B}}\sqrt{\frac{2}{L}}\,[H_2\,(-1)^m - H_1]\int_{y''=-\infty}^{\infty} e^{ivy'}\int_{z'=0}^{B}\cos\,(\psi_n z')\,dz'dy' \\
&+\left\{K_z\frac{A_n}{\sqrt{B}}\cos(\psi_n z')\frac{\partial\bar{\bar{H}}}{\partial z}\right\}\Big|_0^B + \left\{K_z\psi_n\frac{A_n}{\sqrt{B}}\sin(\psi_n z)\bar{\bar{H}}\right\}\Big|_0^B \qquad \textbf{(C.17)} \\
&-\bar{\bar{H}}\,(K_x\beta_m^2 + K_y v^2 + K_2\psi_n^2) = 0
\end{aligned}$$

The third term $K_z \psi_n \frac{A_n}{\sqrt{B}} \sin(\psi_n z) \bar{\bar{H}}$ equates to 0 when integrated from 0 to L.

Substitute the boundary conditions of Equation (C.16) with Equation (C.17) and solve for

$\bar{\bar{\bar{H}}}\ (B_m, v, \psi_n, \infty)$.

$$\bar{\bar{\bar{H}}}(\beta_m, v, \psi_n, \infty) = -\frac{K_x \beta_m A_n \sqrt{\frac{2}{BL}}\ [H_2(-1)^m - H_1] \int_{y'=-\infty}^{\infty} e^{-ivy'} \cdot \int_{z'=0}^{B} \cos(\psi_n z')\ dz'\ dy'}{(K_x \beta_m^2 + K_y v^2 + K_z \psi_n^2)}$$

Return $\bar{\bar{\bar{H}}}$ back to H by multiplying Equation (C.18) by the following three variables: $\frac{A_n}{\sqrt{B}} \cos(\psi_n z)$

$$+ \frac{A_n F \sqrt{\frac{2}{BL}} \int_{y'=y_1}^{y_2} e^{ivy'} \int_{x'=x_1}^{x_2} \sin(\beta_m x')\ dx'\ dy'}{(K_x \beta_m^2 + K_y v^2 + K_z \psi_n^2)} \qquad \textbf{(C.18)}$$

$$+ \frac{A_n I \sqrt{\frac{2}{BL}} \int_{y'=\infty}^{\infty} e^{ivy'} \int_{x'=0}^{L} \sin(\beta_m x')\ dx'\ dy'}{(K_x \beta_m^2 + K_y v^2 + K_z \psi_n^2)}.$$

and sum with respect to n from 0 to ∞; $\frac{e^{-ivy}}{2\pi}$ and integrate with respect to v from $\pm\infty$; $\sqrt{\frac{2}{L}} \sin(\beta_m x)$ and sum with respect to m from 1 to ∞. Upon substitution, Equation (C.18)

reduces to:

$$H(x, y, z, \infty) = \frac{K_x}{LB\pi} \sum_{m=1}^{\infty} \beta_m \sin(\beta_m x)\ [H_2(-1)^m - H_1] \sum_{m=1}^{\infty} A_n^2 \cos(\psi_n z)$$

$$\bullet \int_{z'=o}^{\infty} \cos\,(\psi_n z') \int_{y'=-\infty}^{\infty} \bullet \int_{v=-\infty}^{\infty} \frac{e^{-iv(y-y')}}{(K_x\beta_m^2 + K_y v^2 + K_z\psi_n^2)}\, dv dy'\; dz' \qquad \textbf{(C.19)}$$

$$\frac{F}{BL\pi} \sum_{m=1}^{\infty} \sin(\beta_m x) \sum_{n=0}^{\infty} A_n^2 \cos\,(\psi_n z) \int_{x'=x_1}^{x_2} \sin(\beta_m x') \int_{y'=-y_1}^{y_2} \bullet \int_{v=-\infty}^{\infty} \frac{e^{-iv(y-y')}\, dv dy'\; d}{(K_x\beta_m^2 + K_y v^2 + K_z\psi}$$

$$\frac{I}{BL\pi} \sum_{m=1}^{\infty} \sin(\beta_m x') \sum_{m=1}^{\infty} A_n^2 \cos\,(\psi_n z) \int_{x'=0}^{L} \sin(\beta_m x') \int_{y'=-\infty}^{\infty} \bullet \int_{v=-\infty}^{\infty} \frac{e^{-iv(y-y')}\, dv dy'\; dx'}{(K_x\beta_m^2 + K_y v^2 + K_z\psi_n^2}$$

Note the following integral

$$\int_{v=-\infty}^{\infty} \frac{e^{-iv(y-y')}}{(\beta_m^2 K_x + v_y^2 K_y + \psi_n^2 K_z)}\, dv = \frac{\pi e^{-|y-y'|\frac{\sqrt{\beta_m^2 K_x + \psi_n^2 K_z}}{K_y}}}{\sqrt{K_y(\beta_m^2 K_x + \psi_n^2 K_z)}} \qquad \textbf{(C.20)}$$

Note the following change in variables

$$\begin{array}{ll} n = y - y' \\ d\eta = -dy' \end{array} \qquad \begin{array}{cc} \underline{y'} & \underline{\eta} \\ -\infty & \infty \\ y_1 & y - y_1 \\ y^2 & y - y_2 \\ \infty & -\infty \end{array} \qquad \textbf{(C.21)}$$

Substitute Equations (C.20) and (C.21) into Equation (C.19) and rearrange to get

$$H\;(x, y, z, \infty) = -\frac{K_x}{LB\sqrt{K_y}} \sum_{m=1}^{\infty} \beta_m \sin(\beta_m x)\; [H_2(-1)^m - H_1] \sum_{n=0}^{\infty} \frac{A_n^2 \cos(\psi_n\; z)}{\sqrt{(\beta_m^2 K_x + \psi_n^2 K_z)}}$$

$$\bullet \int_{z'=o}^{B} \cos\ (\psi_n z') \bullet \int_{n=-\infty}^{\infty} \bullet\ e^{-|\eta|\sqrt{\frac{\beta_m^2 K_x + \psi_n^2 K_z}{K_y}}}\ d\eta dz'$$

$$+ \frac{F}{LB\sqrt{K_y}} \sum_{m=1}^{\infty} \sin(\beta_m x) \sum_{n=0}^{\infty} A_n^2 \cos(\psi_n z) \int_{x'=x_1}^{x_2} \frac{\sin(\beta_m x')}{\sqrt{(\beta_m^2 K_x + \psi_n^2 K_z)}} \int_{\eta=y_1-y}^{(y_2-y)} e^{-|\eta|\sqrt{\frac{\beta_m^2 K_x + \psi_n^2 K_z}{K_y}}}\ d\eta dx' \qquad \textbf{(C.22)}$$

$$+ \frac{I}{LB\sqrt{K_y}} \sum_{m=1}^{\infty} \sin(\beta_m x) \sum_{n=0}^{\infty} A_n^2 \cos(\psi_n z) \int_{x'=0}^{L} \frac{\sin(\beta_m x')}{\sqrt{(\beta_m^2 K_x + \psi_n^2 K_z)}} \int_{\eta=-\infty}^{\infty} e^{-|\eta|\sqrt{\frac{\beta_m^2 K_x + \psi_n^2 K_z}{K_y}}}\ d\eta dx'.$$

Note the following integrals:

$$\int_{\eta=-\infty}^{\infty} e^{-a|\eta|}\ d\eta = 2\int_{\eta=0}^{\infty} e^{-a\eta}\ d\eta = \frac{2}{a} \qquad \textbf{(C.23)}$$

$$\int_{\eta=y_1-y}^{y_2-y} e^{-a|\eta|}\ d\eta = [e^{-a|y_1-y|} - 1]\ \frac{sign\ (y_1-y)}{a} - [e^{a|y_2-y|} - 1]\ \frac{sign\ (y_2-y)}{a} \qquad \textbf{(C.24)}$$

$$\text{sign (n)} = \begin{cases} 1 & \text{n is positive} \\ 0 & \text{n is zero} \\ -1 & \text{n is negative} \end{cases}$$

The derivative of Equation (C.24) with respect to y is given by

$$\frac{d}{dy}\left\{ \int_{y_1-y}^{y_2-y} e^{-a|\eta|}\ d\eta \right\} = e^{-a|y_1-y|} - e^{-a|y_2-y|} \qquad \textbf{(C.25)}$$

$$\int_{z'=0}^{B} \cos(\psi_n z')\, dz' = \frac{(\sin(\psi_n B) - \sin(0))}{\psi_n} = \begin{cases} B & n = 0 \\ 0 & n = 1,2,\cdots \end{cases} \quad \textbf{(C.26)}$$

$$\int_{x'=0}^{L} \sin(\beta_m x')\, dx' = \frac{-(\cos(\beta_m L) - \cos(0))}{\beta_m} = \frac{[1-(-1)^m]}{\beta_m} \quad \textbf{(C.27)}$$

$$\int_{x'=x_1}^{x_2} \sin(\beta_m x')\, dx' = \frac{-(\cos(\beta_m x_2) - \cos(\beta_m x_1))}{\beta_m} \quad \textbf{(C.28)}$$

Substitute Equations (C.23)-(C.24), (C.26)-(C.28) into Equation (C.22) and rearrange to get

$$H\ (x, y, z, \infty) = -\frac{2}{L} \sum_{m=1}^{\infty} \frac{\sin(\beta_m x)}{\beta_m} [H_2(-1)^m - H_1]$$

$$+ \frac{2I}{BL} \sum_{m=1}^{\infty} \frac{\sin(\beta_m x)}{\beta_m} [1-(-1)^m] \bullet \sum_{n=0}^{\infty} \frac{A_n^2 \cos(\psi_n z)}{(\beta_m^2 K_x + \psi_n^2 K_z)} \quad \textbf{(C.29)}$$

$$+ \frac{F}{BL} \sum_{m=1}^{\infty} \frac{\sin(\beta_m x)}{\beta_m} [\cos(\beta_m x_1) - \cos(\beta_m x_2)] \bullet \sum_{n=0}^{\infty} \frac{A_n^2 \cos(\psi_n z)}{(\beta_m^2 K_x + \psi_n^2 K_z)}$$

$$\bullet \left\{ [e^{-a|y_1 - y|} - 1]\ sign\ (y_1 - y) - [e^{-a|y_2 - y|} - 1]\ sign\ (y_2 - y) \right\}$$

where

$$a = \frac{\sqrt{\beta_m^2 K_x + \psi_n^2 K_z}}{K_y}$$

$$A_n = \begin{cases} 1 & n = 0 \\ \sqrt{2} & n = 1,2,\cdots \end{cases}$$

z: $\psi_n = \dfrac{n\pi}{B}$ $\quad n = 0,1,\cdots$

x: $\beta_m = \dfrac{n\pi}{L}$ $\quad m = 1,2,\cdots$

$$\sum_{m=1}^{\infty} \frac{\sin(\beta_m x)}{\beta_m} = \frac{(L-x)}{2} \qquad iff\ x > 0 \tag{C.30}$$

$$\sum_{m=1}^{\infty} \frac{\sin(\beta_m x)(-1)^m}{\beta_m} = \frac{-x}{2} \tag{C.31}$$

$$\sum_{m=1}^{\infty} \frac{\sin(\beta_m x)}{\beta_m^3} = \frac{L^2 x}{6} - \frac{L x^2}{4} + \frac{x^3}{12} \qquad iff\ x > 0 \tag{C.32}$$

$$\sum_{m=1}^{\infty} \frac{\sin(\beta_m x)(-1)^m}{\beta_m^3} = -\frac{L^2 x}{12} + \frac{x^3}{12} \tag{C.33}$$

Substitute Equations (C.30)-(C.33) into Equation (C.29):

$$H\ (x, y, z, \infty) = H_1 + (H_2 - H_1)\frac{x}{L} + \frac{I}{2K_x B}(Lx - x^2)$$

$$+ \frac{4I}{LB}\sum_{m=1}^{\infty} \frac{\sin(\beta_m x)}{\beta_m}[1-(-1)^m]\sum_{n=1}^{\infty} \frac{\cos(\psi_n z)}{(\beta_m^2 K_x + \psi_n^2 K_z)}$$

$$+ \frac{F}{BL}\sum_{m=1}^{\infty} \frac{\sin(\beta_m x)}{\beta_m}[\cos(\beta_m x_1) - \cos(\beta_m x_2)]\sum_{n=0}^{\infty} \frac{A_n^2 \cos(\psi_n z)}{(\beta_m^2 K_x + \psi_n^2 K_z)} \tag{C.34}$$

$$\bullet\left\{[e^{-a|y_1-y|} - 1]\ sign(y_1 - y) - [e^{-a|y_2-y|} - 1]\ sign(y_2 - y)\right\}$$

where

$$A_n = \begin{cases} 1 & n = 0 \\ \sqrt{2} & n = 1,2,\cdots \end{cases}$$

$$\psi_n = n\pi/B \qquad n = 0,1,\cdots$$

$$\beta_m = m\pi/L \qquad m = 1,2,\cdots$$

$$a = \frac{\sqrt{\beta_m^2 K_x + \psi_n^2 K_z}}{K_y}$$

and

$$\sum_{m=1}^{\infty} \frac{\cos(\beta_m x)}{(\beta_m^2 + \alpha^2)} = \frac{L\cosh[\alpha L(1 - x/L)]}{2\alpha \sinh[L\alpha]} - \frac{1}{2\alpha^2} \qquad \textbf{(C.35)}$$

where $\alpha^2 = \psi_n^2 K_z / K_x$, cosh[•] is the hyperbolic cosine function, sinh[•] is the hyperbolic sine function and

$$\sum_{m=1}^{\infty} \frac{\cos(\beta_m x)\ (-1)^m}{(\beta_m^2 + \alpha^2)} = \frac{L\cosh[\alpha x]}{2\alpha \sinh[L\alpha]} - \frac{1}{2\alpha^2}. \qquad \textbf{(C.36)}$$

Integrating Equations (C.35) and (C.36) with respect to x gives the following new infinite series:

$$\sum_{m=1}^{\infty} \frac{\sin(\beta_m x)}{\beta_m(\beta_m^2 + \alpha^2)} = \frac{-L\sinh[\alpha L(1 - x/L)]}{2\alpha^2 \sinh[L\alpha]} + \frac{(L - x)}{2\alpha^2} \qquad \textbf{(C.37)}$$

$$\sum_{m=1}^{\infty} \frac{\sin(\beta_m x)(-1)^m}{\beta_m(\beta_m^2 + \alpha^2)} = \frac{L\sinh[\alpha x]}{2\alpha^2 \sinh[\alpha L]} - \frac{x}{2\alpha^2}. \qquad \textbf{(C.38)}$$

Substitute Equations (C.37) and (C.38) into Equation (C.34).

$$H(x, y, z, \infty) = H_1 + (H_2 - H_1)\frac{x}{L} + \frac{I(Lx - x^2)}{2K_x B}$$

$$- \frac{2I}{K_z B} \sum_{n=1}^{\infty} \frac{\cos(\psi_n z)}{\psi_n^2 \sinh(\alpha L)} \left\{ \sinh[\alpha(L - x)] - \sinh(\alpha L) + \sinh(\alpha x) \right\}$$

$$- \frac{F}{BL} \sum_{m=1}^{\infty} \frac{\sin(\beta_m)}{\beta_m} [\cos(\beta_m x_2) - \cos(\beta_m x_1)] \bullet \sum_{n=0}^{\infty} \frac{A_n^2 \cos(\psi_n Z)}{(\beta_m^2 K_x + \psi_n^2 K_z)} \qquad \textbf{(C.39)}$$

$$\bullet \left\{ [e^{-a|y_1 - y|} - 1]\ sign(y_1 - y) - [e^{-a|y_2 - y|} - 1]\ sign(y_2 - y) \right\}$$

where

$$a = \frac{\sqrt{\beta_m^2 K_x + \psi_n^2 K_z}}{K_y}$$

$$\alpha = \psi_n \sqrt{K_z / K_x}$$

$$\beta = m\pi / L \quad m = 1,2,\cdots$$

$$\psi_n = n\pi / B \quad n = 0,1,\cdots$$

Note the following infinite series

$$\sum_{n=1}^{\infty} \frac{\sin(\psi_n z)}{\Psi_n} = \frac{(B - z)}{2} \tag{C.40}$$

$$\sum_{n=1}^{\infty} \frac{\cos(\psi_n z)}{\psi_n^2} = \frac{B^2}{6} - \frac{Bz}{2} - \frac{z^2}{4} \tag{C.41}$$

$$\sum_{n=1}^{\infty} \frac{\cos(\psi_n z)}{(\beta_m^2 K_x + \psi_n^2 K_z)} = \frac{B \cosh[(B - z)\beta_m \sqrt{K_x/K_z}]}{2\beta_m \sqrt{K_x K_z} \sinh[B\beta_m \sqrt{K_x/K_z}]} - \frac{1}{2\beta_m^2 K_x} \tag{C.42}$$

$$\sum_{m=1}^{\infty} \frac{\sin(\beta_m x)}{\beta_m^3} [\cos(\beta_m x_2) - \cos(\beta_m x_1)] = \frac{x(x_2^2 - x_1^2)}{4} - \frac{Lx(x_2 - x_1)}{2} + \frac{L(x - x_1)^2 U(x - x_1)}{4} - \frac{L(x - x_2)^2 U(x - x_2)}{4} \tag{C.43a}$$

$$\sum_{m=1}^{\infty} \frac{\sin(\beta_m x)}{\beta_m(\beta_m^2 + \alpha^2)} [\cos(\beta_m x_2) - \cos(\beta_m x_1)] = \frac{L\sinh[\alpha(L - (x + x_1))] - \sinh[\alpha(L - (x + x_2))]}{4\alpha^2 \sinh[\alpha L]}$$
$$- \frac{L\sinh[\alpha(L - (x - x_2))]U(x - x_2)}{4\alpha^2 \sinh[\alpha L]} + \frac{L\sinh[\alpha(L - (x_2 - x))]U(x_2 - x)}{4\alpha^2 \sinh[\alpha L]}$$
$$+ \frac{L\sinh[\alpha(L - (x - x_1))]U(x - x_1)}{4\alpha^2 \sinh[\alpha L]} - \frac{L\sinh[\alpha(L - (x_1 - x))]U(x_1 - x)}{4\alpha^2 \sinh[\alpha L]} \quad \textbf{(C.43b)}$$
$$- \frac{L}{2\alpha^2} [U(x - x_1) - U(x - x_2)]$$

Substitute Equations (C.40)-(C.44) into Equation (C.39) and rearrange terms to get the final solution.

The steady-state, 3-D hydraulic head is given by:

$$H(x,y,z,\infty) = H_1 + (H_2 - H_1)\frac{x}{L} + \frac{I(Lx - x^2)}{2K_x B} + \frac{I}{K_z B}[\frac{B^2}{3} - Bz + \frac{z^2}{2}]$$
$$- \frac{2I}{K_z B} \sum_{n=1}^{\infty} \frac{\cos(\psi_n z)}{\psi_n^2 \sinh(\alpha L)} \left\{ \sinh[\alpha(L - x)] + \sinh(\alpha x) \right\}$$
$$+ \frac{F}{2K_x B}[sign(y_2 - y) - sign(y_1 - y)][- \frac{x(x_2^2 - x_1^2)}{2L} + x(x_2 - x_1) - \frac{(x - x_1)^2 U(x - x_1)}{2}$$
$$+ \frac{(x - x_2)^2 U(x - x_2)}{2}] \quad \textbf{(C.44)}$$
$$+ \frac{F}{2K_z B}[sign(y_2 - y) - sign(y_1 - y)][U(x - x_1) - U(x - x_2)][\frac{B^2}{3} - Bz + \frac{z^2}{2}]$$
$$- \frac{F}{2K_z B}[sign(y_2 - y) - sign(y_1 - y)] \sum_{n=1}^{\infty} \frac{\cos(\psi_n z)}{\psi_n^2 \sinh(\alpha L)} \Big\{ \sinh[\alpha(L - (x + x_1))]$$
$$- \sinh[\alpha(L - (x + x_2))] - \sinh[\alpha(L - (x - x_2))]U(x - x_2)$$
$$+ \sinh[\alpha(L - (x_2 - x))]U(x_2 - x) + \sinh[\alpha(L - (x - x_1))]U(x - x_1)$$
$$- \sinh[\alpha(L - (x_1 - x))]U(x_1 - x) \Big\}$$

$$-\frac{F}{LBK_x}\sum_{m=1}^{\infty}\frac{\sin(\beta_m x)}{\beta_m^3}[\cos(\beta_m x_2)-\cos(\beta_m x_1)]\left[e^{-\beta_m\sqrt{\frac{K_x}{K_y}}\,|y_1-y|}\bullet sign(y_1-y_2)\right.$$
$$\left.-e^{-\beta_m\sqrt{\frac{K_x}{K_y}}\,|y_2-y|}\bullet\ sign(y_2-y)\right]$$

$$+\frac{2F}{LB}\sum_{n=1}^{\infty}\psi_n\sin(\psi_n z)\sum_{m=1}^{\infty}\frac{\sin(\beta_m x)}{\beta_m(\beta_m^2K_x+\psi_n^2K_z)}[\cos(\beta_m x_2)-\cos(\beta_m x_1)]$$
$$\bullet[e^{-a|y_i-y|}\bullet sign(y_1-y)-e^{-a|y_2-y|}\bullet sign(y_2-y)]$$

where

β_m = $m\pi/L$ m =1, 2, $\cdots$

ψ_n = $n\pi/L$ n = 0, 1, $\cdots$

$$a=\frac{\sqrt{\beta_m^2K_x+\psi_n^2K_z}}{K_y}\ ,\qquad \alpha=\psi_n\sqrt{K_z/K_x}$$

The spatial derivatives of the hydraulic head are computed as:

$$\frac{\partial H}{\partial x}(x,y,z,\infty)=\frac{(H_2-H_1)}{L}+\frac{I(L-2x)}{2K_xB}$$
$$-\frac{2I}{B\sqrt{K_xK_z}}\sum_{n=1}^{\infty}\frac{\cos(\psi_n z)}{\psi_n\sinh(\alpha L)}\left\{-\cosh[\alpha(L-x)]+\cosh(\alpha x)\right\}$$
$$+\frac{F}{2K_xB}[sign(y_2-y)-sign(y_1-y)][-\frac{x(x_2^2-x_1^2)}{2L}+(x_2-x_1)-(x-x_1)U(x-x_1)$$
$$+(x-x_2)U(x-x_2)] \qquad \textbf{(C.45)}$$
$$-\frac{F}{2B\sqrt{K_xK_z}}[sign(y_2-y)-sign(y_1-y)]\sum_{n=1}^{\infty}\frac{\cos(\psi_n z)}{\psi_n\sinh(\alpha L)}[-\cosh[\alpha(L-(x+x_1))]$$
$$+\cosh[\alpha(L-(x+x_2))]+\cosh[\alpha(L-(x-x_2))]U(x-x_2)$$
$$+\cosh[\alpha(L-(x_2-x))]U(x_2-x)-\cosh[\alpha(L-(x-x_1))]U(x-x_1)$$
$$-\cosh[\alpha(L-(x_1-x))]U(x_1-x)]$$

$$
-\frac{F}{LBK_x}\sum_{m=1}^{\infty}\frac{\cos(\beta_m x)}{\beta_m^2}[\cos(\beta_m x_2)-\cos(\beta_m x_1)]\left[sign(y_1-y)\bullet e^{-\beta_m|y_1-y|\sqrt{\frac{K_x}{K_y}}}\right.
$$

$$
\left.-sign(y_2-y)\bullet e^{-\beta_m|y_2-y|\sqrt{\frac{K_x}{K_y}}}\right]
$$

$$
-\frac{2F}{LB}\sum_{n=1}^{\infty}\cos(\psi_n z)\sum_{m=1}^{\infty}\frac{\sin(\beta_m x)}{\beta_m\sqrt{K_y(\beta_m^2 K_x+\psi_n^2 K_z)}}[\cos(\beta_m x_2)-\cos(\beta_m x_1)]
$$

$$
\bullet\left\{e^{-a|y_1-y|}-e^{-a|y_2-y|}\right\}
$$

$$
\frac{\partial H}{\partial y}(x,y,z,\infty) = -\frac{F}{LB\sqrt{K_x K_y}}\sum_{m=1}^{\infty}\frac{\sin(\beta_m x)}{\beta_m^2}[\cos(\beta_m x_2)-\cos(\beta_m x_1)]
$$

$$
\bullet\left[e^{-\beta_m|y_1-y|\sqrt{\frac{K_x}{K_y}}}-e^{-\beta_m|y_2-y|\sqrt{\frac{K_x}{K_y}}}\right] \qquad \textbf{(C.46)}
$$

$$
-\frac{2F}{LB}\sum_{n=1}^{\infty}\cos(\psi_n z)\sum_{m=1}^{\infty}\frac{\sin(\beta_m x)}{\beta_m(\beta_m^2 K_x+\psi_n^2 K_z)}[\cos(\beta_m x_2)-\cos(\beta_m x_1)]
$$

$$
\bullet\left\{e^{-a|y_1-y|}sign(y_1-y)-e^{-a|y_2-y|}sign(y_2-y)\right\}
$$

$$\frac{\partial H}{\partial z}(x,y,z,\infty) = \frac{I}{K_z B}(-B+z) + \frac{2I}{K_z B}\sum_{n=1}^{\infty}\frac{\sin(\psi_n z)}{\psi_n \sinh(\alpha L)}\left\{\sinh[\alpha(L-x)]\right.$$

$$\left. + \sinh(\alpha x)\right\} + \frac{F}{2K_z B}[sign(y_2-y) - sign(y_1-y)][U(x-x_1) - U(x-x_2)][-B+z]$$

$$+ \frac{F}{2K_z B}[sign(y_2-y) - sign(y_1-y)]\sum_{n=1}^{\infty}\frac{\sin(\psi_n z)}{\psi_n \sinh(\alpha L)}[\sinh[\alpha(L-(x+x_1))] \tag{C.47}$$

$$- \sinh[\alpha(L-(x+x_2))] - \sinh[\alpha(L-(x-x_2))]U(x-x_2)$$

$$+ \sinh[\alpha(L-(x_2-x))]U(x_2-x) + \sinh[\alpha(L-(x-x_1))]U(x-x_1)$$

$$- \sinh[\alpha(L-(x_1-x))]U(x_1-x)$$

$$+ \frac{2F}{LB}\sum_{n=1}^{\infty}\psi_n \sin(\psi_n z)\sum_{m=1}^{\infty}\frac{\sin(\beta_m x)}{\beta_m(\beta_m^2 K_x + \psi_n^2 K_z)}[\cos(\beta_m x_2) - \cos(\beta_m x_1)]$$

$$\bullet[e^{-a|y_i - y|} \bullet sign(y_1-y) - e^{-a|y_2-y|} \bullet sign(y_2-y)]$$

Special 2-D Areal Case

$$K_z \to 0$$

The 2-D areal groundwater mounding problem can be computed by integrating the partial differential Equation given in Equation (C.1) with respect to z between 0 and B:

$$K_x \frac{\partial^2 \overline{H}}{\partial x^2}B + K_y \frac{\partial^2 \overline{H}}{\partial y^2}B + \left[K_z \frac{\partial H}{\partial z}\right]\Big|_0^B = 0 \tag{C.48}$$

where

$$\overline{H}(x,y,\infty) = \int_0^B \frac{H(x,y,z,\infty)}{B}dz \tag{C.49}$$

is the depth averaged head.

Substitute the boundary conditions of Equation (C.2) into Equation (C.48) and then divide through by B:

$$K_x \frac{\partial^2 \overline{H}}{\partial x^2} + K_y \frac{\partial^2 \overline{H}}{\partial y^2} + \frac{I}{B} + \frac{F}{B}[U(x-x_1) - U(x-x_2)][U(y-y_1) - U(y-y_2)] = 0 \quad \textbf{(C.50)}$$

with boundary conditions

$$\begin{aligned} \overline{H}(0,y,\infty) &= H_1 \\ \overline{H}(L,y,\infty) &= H_2 \\ \frac{\partial \overline{H}}{\partial y}(x,\pm\infty,\infty) &= 0 \end{aligned} \quad \textbf{(C.51)}$$

The analytical solution given by Equation (C.44) can also be integrated with respect to *z* and then divide by *B* to give the depth averaged head. This corresponds to the solution of Equations (C.50)-(C.51).

Note the following integrals

$$\frac{1}{B}\int_0^B \left(\frac{B^2}{3} - Bz + \frac{z^2}{2}\right)dz = \frac{1}{B}\left[\frac{B^2 z}{3} - \frac{Bz^2}{2} + \frac{z^3}{6}\right]\Big|_0^B = 0 \quad \textbf{(C.52)}$$

$$\frac{1}{B}\int_0^B \cos(\psi_n z)\,dz = \frac{[\sin(\psi_n B) - \sin(\psi_n 0)]}{B\psi_n} = \begin{cases} 1 & \text{if } \psi_n = 0 \\ 0 & \text{if } \psi_n > 0 \end{cases} \quad \textbf{(C.53)}$$

Substitute Equations (C.52)-(C.53) into Equation (C.44) after integrating Equation (C.44) with respect to *z* and dividing by B. Also substitute in the infinite series solution given by Equation (C.43a). The 2-D, steady-state hydraulic head is computed as:

$$\overline{H}(x,y,\infty) = H_1 + (H_2 - H_1)\frac{x}{L} + \frac{I(Lx - x^2)}{2K_xB}$$
$$+ \frac{F}{2K_xB}[sign(y_2 - y) - sign(y_1 - y)][-\frac{x(x_2^2 - x_1^2)}{2L} + x(x_2 - x_1) - \frac{(x - x_1)^2U(x - x_1)}{2}$$
$$+ \frac{(x - x_2)^2U(x - x_2)}{2}] - \frac{F}{BLK_x}\sum_{m=1}^{\infty}\frac{\sin(\beta_m x)}{\beta_m^3}[\cos(\beta_m x_2) - \cos(\beta_m x_1)] \quad \textbf{(C.54)}$$
$$\bullet\left[sign(y_1 - y)\bullet e^{-\beta_m|y_1 - y|\sqrt{\frac{K_x}{K_y}}} - sign(y_2 - y)\bullet e^{-\beta_m|y_2 - y|\sqrt{\frac{K_x}{K_y}}}\right]$$

where

$$\beta_m = \pi m/L \quad \textbf{(C.55)}$$

and the spatial derivatives of the hydraulic head

$$\frac{\partial\overline{H}}{\partial x}(x,y,\infty) = \frac{(H_2 - H_1)}{L} + \frac{I(L - 2x)}{2K_xB}$$
$$+ \frac{F}{2K_xB}[sign(y_2 - y) - sign(y_1 - y)][-\frac{(x_2^2 - x_1^2)}{2L} + (x_2 - x_1) - (x - x_1)U(x - x_1)$$
$$+ (x - x_2)U(x - x_2)]$$
$$- \frac{F}{BLK_x}\sum_{m=1}^{\infty}\frac{\cos(\beta_m x)}{\beta_m^2}[\cos(\beta_m x_2) - \cos(\beta_m x_1)] \quad \textbf{(C.56)}$$
$$\bullet\left[sign(y_1 - y)\bullet e^{-\beta_m|y_1 - y|\sqrt{\frac{K_x}{K_y}}} - sign(y_2 - y)\bullet e^{-\beta_m|y_2 - y|\sqrt{\frac{K_x}{K_y}}}\right]$$

$$\frac{\partial\overline{H}}{\partial y}(x,y,\infty) = -\frac{F}{BL\sqrt{K_xK_y}}\sum_{m=1}^{\infty}\frac{\sin(\beta_m x)}{\beta_m^2}[\cos(\beta_m x_2) - \cos(\beta_m x_1)]$$
$$\bullet\left[e^{-\beta_m|y_1 - y|\sqrt{\frac{K_x}{K_y}}} - e^{-\beta_m|y_2 - y|\sqrt{\frac{K_x}{K_y}}}\right]. \quad \textbf{(C.57)}$$

Special 2-D x, z Cross-Sectional Case

$$K_y \to 0$$
$$y_2 \to +\infty$$
$$y_1 \to -\infty$$

The 2-D cross-sectional groundwater mounding problem for the *x*, *z* domain can be computed by setting $K_y = 0$ and $y_z = +\infty$ and $y_1 = -\infty$ in Equation (C.44):

$$\begin{aligned}
\hat{H}(x,z,\infty) &= H_1 + (H_2 - H_1)\frac{x}{L} + \frac{I(Lx - x^2)}{2K_xB} + \frac{I}{K_zB}\left(\frac{B^2}{3} - Bz + \frac{z^2}{2}\right) \\
&- \frac{2I}{K_zB}\sum_{n=1}^{\infty}\frac{\cos(\psi_n z)}{\psi_n^2\sinh(\alpha L)}\left\{\sinh[\alpha(L-x)] + \sinh(\alpha x)\right\} \\
&+ \frac{F}{K_xB}\left[-\frac{x(x_2^2 - x_1^2)}{2L} + x(x_2 - x_1) - \frac{(x-x_1)^2U(x-x_1)}{2} + \frac{(x-x_2)^2U(x-x_2)}{2}\right] \\
&+ \frac{F}{K_zB}\left[U(x-x_1) - U(x-x_2)\right]\left[\frac{B^2}{3} - Bz + \frac{z^2}{2}\right] \\
&- \frac{F}{K_zB}\sum_{n=1}^{\infty}\frac{\cos(\psi_n z)}{\psi_n^2\sinh(\alpha L)}\Big\{\sinh[\alpha(L-(x+x_1))] - \sinh[\alpha(L-(x+x_2))] \\
&- \sinh[\alpha(L-(x-x_2))]U(x-x_2) + \sinh[\alpha(L-(x_2-x))]U(x_2-x) \\
&+ \sinh[\alpha(L-(x-x_1))]U(x-x_1) - \sinh[\alpha(L-(x_1-x))]U(x_1-x)\Big\}
\end{aligned} \tag{C.58}$$

where

β_m = mπ/L
ψ_n = nπ/B
α = $\psi_n\sqrt{K_z/K_x}$

$$
\begin{aligned}
\frac{\partial \hat{H}}{\partial x}(x,z,\infty) &= \frac{(H_2 - H_1)}{L} + \frac{I(L-2x)}{2K_xB} \\
&- \frac{2I}{B\sqrt{K_xK_z}} \sum_{n=1}^{\infty} \frac{\cos(\psi_n z)}{\psi_n \sinh(\alpha L)} [-\cosh[\alpha(L-x) + \cosh(\alpha x)] \\
&+ \frac{F}{K_xB}[-\frac{(x_2^2 - x_1^2)}{2L} + (x_2 - x_1) - (x-x_1)U(x-x_1) + (x-x_2)U(x-x_2)] \\
&- \frac{F}{B\sqrt{K_xK_z}} \sum_{n=1}^{\infty} \frac{\cos(\psi_n z)}{\psi_n \sinh(\alpha L)} \Big\{ -\cosh[\alpha(L-(x+x_1))] + \cosh[\alpha(L-(x+x_2))] \\
&+ \cosh[\alpha(L-(x-x_2))]U(x-x_2) + \cosh[\alpha(L-(x_2-x))]U(x-x_2) \\
&- \cosh[\alpha(L-(x-x_1))]U(x-x_1) - \cosh[\alpha(L-(x_1-x))]U(x_1-x) \Big\}
\end{aligned}
\tag{C.59}
$$

$$
\begin{aligned}
\frac{\partial \hat{H}}{\partial z}(x,z,\infty) &= \frac{I}{K_zB}(-B+z) + \frac{F}{K_zB}[U(x-x_1) - U(x-x_2)][(-B+z)] \\
&+ \frac{2I}{K_zB} \sum_{n=1}^{\infty} \frac{\sin(\psi_n z)}{\psi_n \sinh(\alpha L)} \Big\{ \sinh[\alpha(L-x)] + \sinh(\alpha x) \Big\} \\
&+ \frac{F}{K_zB} \sum_{n=1}^{\infty} \frac{\sin(\psi_n z)}{\psi_n \sinh(\alpha L)} \Big\{ \sinh[\alpha(L-(x+x_1))] - \sinh[\alpha(L-(x+x_2))] \\
&- \sinh[\alpha(L-(x-x_2))]U(x-x_2) + \sinh[\alpha(L-(x_2-x))]U(x_2-x) \\
&+ \sinh[\alpha(L-(x-x_1))]U(x-x_1) - \sinh[\alpha(L-(x_1-x))]U(x_1-x) \Big\}
\end{aligned}
\tag{C.60}
$$

Special 1-D Longitudinal Case

$$
\begin{aligned}
K_y &\to 0 \\
y_2 &\to +\infty \\
y_1 &\to -\infty \\
K_z &\to 0
\end{aligned}
$$

The 1-D longitudinal groundwater mounding problem can be found by setting $K_y = 0$ and $y_2 = +\infty$, $y_1 = -\infty$ in Equation (C.55).

$$H^*(x,\infty) = H_1 + (H_2 - H_1)\frac{x}{L} + \frac{I(Lx - x^2)}{2K_xB}$$

$$- \frac{2F}{BLK_x}\left[\frac{x(x_2^2 - x_1^2)}{4} - \frac{Lx(x_2 - x_1)}{2} + \frac{L(x - x_1)^2U(x - x_1)}{4} - \frac{L(x - x_2)^2U(x - x_2)}{4}\right] \quad \textbf{(C.61)}$$

$$\frac{\partial H^*}{\partial x}(x,\infty) = \frac{(H_2 - H_1)}{L} + \frac{I(L - 2x)}{2K_xB}$$

$$- \frac{2F}{BLK_x}\left\{\frac{(x_2^2 - x_1^2)}{4} - \frac{L(x_2 - x_1)}{2} + \frac{L(x - x_1)U(x - x_1)}{2} - \frac{L(x - x_2)U(x - x_2)}{2}\right\} \quad \textbf{(C.62)}$$

Special Depth Averaged Solution

$$h(x,y,\infty) = \frac{\int_{z_1}^{z_2} H(x,y,z,\infty)dz}{\int_{z_1}^{z_2} dz}$$

The 3-D solutions given by Equations (C.44)-(C.47) can be depth averaged by integrating each equation with respect to z from z_1 to z_2 and then dividing through by z_2 - z_1 (where $B \geq z_2 > z_1 \geq 0$). The resultant values will represent the average value of the variable over the depth interval z_1 to z_2. The following new variables are defined as:

$$h(x,y,\infty) = \int_{z_1}^{z_2} \frac{H(x,y,z,\infty)}{(z_2 - z_1)}\, dz \quad \textbf{(C.63)}$$

$$\frac{\partial h}{\partial x}(x,y,\infty) = \int_{z_1}^{z_2} \frac{\partial H(x,y,z,\infty)}{\partial x} \bullet \frac{1}{(z_2 - z_1)}\, dz \quad \textbf{(C.64)}$$

$$\frac{\partial h}{\partial y}(x,y,\infty) = \int_{z_1}^{z_2} \frac{\partial H(x,y,z,\infty)}{\partial y} \bullet \frac{1}{(z_2 - z_1)}\, dz \quad \textbf{(C.65)}$$

$$\frac{\partial h}{\partial z}(x,y,\infty) = \int_{z_1}^{z_2} \frac{\partial H(x,y,z,\infty)}{\partial z} \bullet \frac{1}{(z_2 - z_1)}\, dz \qquad \textbf{(C.66)}$$

Upon integration, Equations (C.44) - (C.47) reduce to:

$$\begin{aligned}
h(x,y,\infty) = {} & H_1 + (H_2 - H_1)\frac{x}{L} + \frac{I(Lx - x^2)}{2K_xB} + \frac{I}{K_zB} \\
& \left(\frac{B^2}{3} - \frac{B(z_2 + z_1)}{2} + \frac{(z_2^2 + z_1z_2 + z_1^2)}{6}\right) \qquad \textbf{(C.67)} \\
& - \frac{2I}{K_zB}\sum_{n=1}^{\infty}\frac{[\sin(\psi_nz_2) - \sin(\psi_nz_1)]}{(z_2 - z_1)\psi_n^3\sinh(\alpha L)}[\sinh[\alpha(L - x)] + \sinh(\alpha x)] \\
& + \frac{F}{2K_zB}[sign(y_2 - y) - sign(y_1 - y)][U(x - x_1) - U(x - x_2)] \\
& \left[\frac{B^2}{3} - \frac{B(z_2 + z_1)}{2} + \frac{(z_2^2 + z_1z_2 + z_1^2)}{6}\right] \\
& - \frac{F}{2K_zB}[sign(y_2 - y) - sign(y_1 - y)]\sum_{n=1}^{\infty}\frac{[\sin(\psi_nz_2) - \sin(\psi_nz_1)]}{\psi_n^3(z_z - z_1)\sinh(\alpha L)}\Big\{\sinh[\alpha(L - (x + x_1))] \\
& - \sinh[\alpha(L - (x + x_2))] - \sinh[\alpha(L - (x - x_2))]U(x - x_2) \\
& + \sinh[\alpha(L - (x_2 - x))]U(x_2x) + \sinh[\alpha(L - (x - x_1))]U(x - x_1) \\
& - \sinh[\alpha(L - (x_1 - x))]U(x_1 - x)\Big\} \\
& - \frac{F}{LBK_x}\sum_{m=1}^{\infty}\frac{\sin(\beta_mx)}{\beta_m^3}[\cos(\beta_mx_1) - \cos(\beta_mx_1)]\bullet\Bigg[sign(y_1 - y)\bullet e^{-\beta_m|y_1 - y|\sqrt{\frac{K_x}{K_y}}} \\
& - sign(y_2 - y)\bullet e^{-\beta_m|y_2 - y|\sqrt{\frac{K_x}{K_y}}}\Bigg]
\end{aligned}$$

$$-\frac{2F}{LB}\sum_{n=1}^{\infty}\frac{[\sin(\psi_n z_2)-\sin(\psi_n z_1)]}{\psi_n(z_2-z_1)}\sum_{m=1}^{\infty}\frac{\sin(\beta_m x)}{\beta_m(\beta_m^2 K_x+\psi_n^2 K_z)}[\cos(\beta_m x_2)-\cos(\beta_m x_1)]$$

$$\bullet[e^{-a|y_1-y|}sign(y_1-y) - e^{-a|y_2-y|}sign(y_2-y)]$$

where

β_m = mπ/L

ψ_n = nπ/B

$$a = \sqrt{\frac{\beta_m^2 K_x+\psi_n^2 K_z}{K_y}}$$

$$\alpha = \psi_n\sqrt{K_z/K_x}$$

$$\frac{\partial h}{\partial x}(x,y,\infty) = \frac{(H_2-H_1)}{L} + \frac{I(L-2x)}{2K_xB}$$

1

$$-\frac{2I}{B\sqrt{K_xK_z}}\sum_{n=1}^{\infty}\frac{[\sin(\psi_n z_2)-\sin(\psi_n z_1)]}{\psi_n^2(z_2-z_1)\sinh(\alpha L)}\left\{-\cosh[\alpha(L-x)]+\cosh(\alpha x)\right\} \tag{C.68}$$

$$+\frac{F}{2K_xB}[sign(y_2-y)-sign(y_1-y)][-\frac{(x_2^2-x_1^2)}{2L} + (x_2-x_1) - (x-x_1)U(x-x_1)$$

$$+(x-x_2)U(x-x_2)]$$

$$\frac{\partial h}{\partial y}(x,y,\infty) = -\frac{F}{LB\sqrt{K_xK_y}}\sum_{m=1}^{\infty}\frac{\sin(\beta_m x)}{\beta_m^2}[\cos(\beta_m x_2)-\cos(\beta_m x_1)] \tag{C.69}$$

$$-\frac{2F}{LB}\sum_{n=1}^{\infty}\frac{[\sin(\psi_n z_2)-\sin(\psi_n z_1)]}{\psi_n(z_2-z_1)}\sum_{m=1}^{\infty}\frac{\cos(\beta_m x)}{(\beta_m^2 K_x+\psi_n^2 K_z)}[\cos(\beta_m x_2)-\cos(\beta_m x_1)]$$

$$\bullet\left\{e^{-a|y_1-y|}sign(y_1-y)-e^{-a|y_2-y|}sign(y_2-y)\right\}$$

$$-\frac{F}{LBK_x}\sum_{m=1}^{\infty}\frac{\cos(\beta_m x)}{B_m^2}[\cos(\beta_m x_2)]-\cos(\beta_m x_1)]\left[sign(y_1-y)\bullet e^{-\beta_m|y_1-y|\sqrt{\frac{K_x}{K_y}}}\right.$$

$$\left.-sign(y_2-y)\bullet e^{-\beta_m|y_2-y|\sqrt{\frac{K_x}{K_y}}}\right]$$

$$-\frac{2F}{LB}\sum_{n=1}^{\infty}\frac{[\cos(\psi_n z_2)-\cos(\psi_n z_1)]}{(z_2-z_1)}\sum_{m=1}^{\infty}\frac{\sin(\beta_m x)}{\beta_m(\beta_m^2 K_x+\psi_n^2 K_z)}[\cos(\beta_m x_2)-\cos(\beta_m x_1)]$$

$$\bullet\left\{e^{-a|y_1-y|}\bullet sign(y_1-y)-e^{-a|y_2-y|}\bullet sign(y_2-y)\right\}$$

$$\bullet\left[e^{-\beta_m|y_1-y|\sqrt{\frac{K_x}{K_y}}}-e^{-\beta_m|y_2-y|\sqrt{\frac{K_x}{K_y}}}\right]$$

$$-\frac{2F}{LB}\sum_{n=1}^{\infty}\frac{[\sin(\psi_n z_2)-\sin(\psi_n z_1)]}{\psi_n(z_2-z_1)}\sum_{m=1}^{\infty}\frac{\sin(\beta_m x)}{\beta_m\sqrt{K_y(\beta_m^2 K_x+\psi_n K_z)}}$$

$$\bullet[\cos(\beta_m x_2)-\cos(\beta_m x_1)]\left\{e^{-a|y_1-y|}-e^{-a|y_2-y|}\right\}$$

$$+\frac{F}{2K_xB}[sign(y_2-y)-sign(y_1-y)]\left[-\frac{x(x_2^2-x_1^2)}{2L}+x(x_2-x_1)-\frac{(x-x_1)^2U(x-x_1)}{2}\right.$$

$$\left.+\frac{(x-x_2)^2U(x-x_2)}{2}\right]$$

$$
\begin{aligned}
\frac{\partial h}{\partial z}(x,y,\infty) \;=\; & \frac{I}{K_z B}\left(-B + \frac{(z_2+z_1)}{2}\right) \\
& - \frac{2I}{K_z B}\sum_{n=1}^{\infty}\frac{[\cos(\psi_n z_2) - \cos(\psi_n z_1)]}{\psi_n^2 (z_2 - z_1)\sinh(\alpha L)}\left\{\sinh[\alpha(L-x)] + \sinh(\alpha x)\right\} \\
& + \frac{F}{2K_z B}[sign(y_2 - y) - sign(y_1 - y)][U(x-x_1) - U(x-x_2)]\left[-B + \frac{(z_2+z_1)}{2}\right] \\
& - \frac{F}{2K_z B}\,[sign(y_2 - y) - sign(y_1 - y)]\sum_{n=1}^{\infty}\frac{[\cos(\psi_n z_2) - \cos(\psi_n z_1)]}{\psi_n^2 (z_2 - z_1)\sinh(\alpha L)} \\
& \cdot\Big\{\sinh[\alpha(L-(x+x_1))] - \sinh[\alpha(L-(x+x_2))] - \sinh[\alpha(L-(x-x_2))]\,U(x-x_2) \\
& + \sinh[\alpha(L-(x_2-x))]\,U(x_2-x) + \sinh[\alpha(L-(x-x_1))]\,U(x-x_1) \\
& - \sinh[\alpha(L-(x_1-x))]\,U(x_1-x)\Big\}
\end{aligned} \tag{C.70}
$$

$$
\begin{aligned}
& - \frac{F}{2B\sqrt{K_x K_z}}[sign(y_2 - y) - sign(y_1 - y)]\sum_{n=1}^{\infty}\frac{[\sin(\psi_n z_2) - \sin(\psi_n z_1)]}{\psi_n^2 (z_2 - z_1)\sinh(\alpha L)} \\
& \Big\{-\cosh[\alpha(L-(x+x_1))]\cosh[\alpha(L-(x+x_2))] + \cosh[\alpha(L-(x-x_2))]\,U(x-x_2) \\
& + \cosh[\alpha(L-(x_2-x))]\,U(x_2-x) - \cosh[\alpha(L-(x-x_1))]\,U(x-x_1) \\
& - \cosh[\alpha(L-(x_1-x))]\,U(x_1-x)\Big\}
\end{aligned}
$$

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APPENDIX D

VERIFICATION AND VALIDATION OF THE EPA'S COMPOSITE MODEL FOR TRANSFORMATION PRODUCTS (EPACMTP), AND ITS DERIVATIVES

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